LIE ALGEBROIDS AND POISSON-NIJENHUIS STRUCTURES [1]

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Abstract

Poisson-Nijenhuis structures for an arbitrary Lie algebroid are defined and studied by means of complete lifts of tensor fields.

1. INTRODUCTION

In our previous paper [4], certain definitions and constructions of graded Lie brackets and lifts of tensor fields over a manifold were generalized to arbitrary Lie algebroids. Since Poisson-Nijenhuis structures seem to fit very well to the Lie algebroid language and, as it was recently shown by Kosmann-Schwarzbach in [5], they give examples of Lie bialgebroid structures in the sense of Mackenzie and Xu [9], we would like to present in this note a Lie algebroid approach to Poisson-Nijenhuis structures.

We start with the definition of a pseudo-Lie algebroid structure on a vector bundle E, as a slight generalization of the notion of a Lie algebroid and we show that such structures are determined by special tensor fields Λ on the dual bundle E^*.

Then, we define the complete lift $\mathrm{d}^{\Lambda}_{\mathsf{T}}$, which reduces to the classical tangent lift d_{T} in the case of the tangent bundle $E = \mathsf{T}M$. We prove that, when we start from $P \in \Gamma(M, \wedge^2 E)$, the complete lift $\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)$ corresponds to a bracket on sections of E^*, which, in the classical case, is the Fuchssteiner-Koszul bracket on 1-forms. In the case of a Lie algebroid over a single point, $\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)$ is closely related to the modified Yang-Baxter equation. Deforming the Lie algebroid bracket by a (1,1) tensor N, we find the corresponding bivector field Λ_N on E^*. Assuming some compatibility conditions for N and P, we can define a Poisson-Nijenhuis structure for a Lie algebroid which provides a whole list of Lie bialgebroid structures.

This unified approach to Poisson and Nijenhuis structures, including the classical case as well, as the case of a real Lie algebra, makes possible to understand common aspects of the theory, which were previously seen separately for different models.

2. TANGENT LIFTS FOR PRE-LIE ALGEBROIDS

Let M be a manifold and let $\tau\colon E \to M$ be a vector bundle. By $\Phi(\tau)$ we denote the graded exterior algebra generated by sections of τ: $\Phi(\tau) = \oplus_{k\in\mathbb{Z}}\Phi^k(\tau)$, where $\Phi^k(\tau) = \Gamma(M, \wedge^k E)$ for $k \geqslant 0$ and $\Phi^k(\tau) = \{0\}$ for $k < 0$. Elements of $\Phi^0(\tau)$ are functions on M, i.e., sections of the bundle $\wedge^0 E = M \times \mathbb{R}$. Similarly, by $\otimes(\tau)$ we denote the tensor algebra $\otimes(\tau) = \oplus_{k\in\mathbb{Z}} \otimes^k (\tau)$, where $\otimes^k(\tau) = \Gamma(M, \otimes^k_M E)$. The

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dual vector bundle we denote by $\pi\colon E^* \to M$. For the tangent bundle $\tau_M\colon \mathsf{T}M \to M$, $\Phi(\tau_M)$ is the exterior algebra of multivector fields, and for the cotangent bundle $\pi_M\colon \mathsf{T}^*M \to M$, we get $\Phi(\pi_M)$, the exterior algebra of differential forms on M.

The cotangent bundle is endowed with the canonical symplectic form ω_M and the corresponding canonical Poisson tensor Λ_M.

Definition 2.1 *A* pseudo-Lie algebroid structure *on a vector bundle $\tau\colon E \to M$ is a bracket (bilinear operation) $[\,,\,]$ on the space $\Phi^1(\tau) = \Gamma(M, E)$ of sections of τ and vector bundle morphisms $\alpha_l, \alpha_r\colon E \to \mathsf{T}M$ (called the left- and right-anchor, respectively), such that*

$$[fX, gY] = f\alpha_l(X)(g)Y - g\alpha_r(Y)(f)X + fg[X, Y] \tag{2.1}$$

for all $X, Y \in \Gamma(E)$ and $f, g \in C^\infty(M)$.

A pseudo-Lie algebroid, with a skew-symmetric bracket $[\,,\,]$ (in this case the left and right anchors coincide), is called a *pre-Lie algebroid*.

A pre-Lie algebroid is called a *Lie algebroid* if the bracket $[\,,\,]$ satisfies the Jacobi identity, i.e., if it provides $\Phi^1(\tau)$ with a Lie algebra structure.

In the following, we establish a correspondence between pseudo-Lie algebroid structures on E and 2-contravariant tensor fields on the bundle manifold E^* of the dual vector bundle $\pi\colon E^* \to M$. Let $X \in \Phi^1(\tau)$. We define a function $\iota_{E^*}X$ on E^* by the formula

$$E^* \ni a \mapsto \iota_{E^*}X(a) = \langle X(\pi(a)), a\rangle,$$

where $\langle\,,\,\rangle$ is the canonical pairing between E and E^*.

Let $\Lambda \in \Gamma(E, \mathsf{T}E \otimes_E \mathsf{T}E)$ be a 2-contravariant tensor field on E. We say that Λ is *linear* if, for each pair (μ, ν) of sections of π, the function $\Lambda(\mathrm{d}\iota_E\mu, \mathrm{d}\iota_E\nu)$, defined on E, is linear.

For each 2-contravariant tensor Λ, we define a bracket $\{\,,\,\}_\Lambda$ of functions by the formula

$$\{f, g\}_\Lambda = \Lambda(\mathrm{d}f, \mathrm{d}g).$$

Theorem 2.1 *For every pseudo-Lie algebroid structure on $\tau\colon E \to M$, with the bracket $[\,,]$ and anchors α_l, α_r, there is a unique 2-contravariant linear tensor field Λ on E^* such that*

$$\iota_{E^*}[X, Y] = \{\iota_{E^*}X, \iota_{E^*}Y\}_\Lambda \tag{2.2}$$

and

$$\pi^*\left(\alpha_l(X)(f)\right) = \{\iota_{E^*}X, \pi^*f\}_\Lambda, \quad \pi^*\left(\alpha_r(X)(f)\right) = \{\pi^*f, \iota_{E^*}X\}_\Lambda, \tag{2.3}$$

for all $X, Y \in \Phi^1(\tau)$ and $f \in C^\infty(M)$.

Conversely, every 2-contravariant linear tensor field Λ on E^* defines a pseudo-Lie algebroid on E by the formulae 2.2 and 2.3.

The pseudo-Lie algebroid structure on E is a pre-Lie algebroid structure (resp. a Lie algebroid structure) if and only if the tensor Λ is skew-symmetric (resp. Λ is a Poisson tensor).

Proof. We shall use local coordinates. Let (x^a) be a local coordinate system on M and let $e_1, \ldots, e_n$ be a basis of local sections of E. We denote by $e^{*1}, \ldots, e^{*n}$ the dual basis of local sections of E^* and by (x^a, y^i) (resp. (x^a, ξ_i)) the corresponding coordinate system on E (resp. E^*), i.e., $\iota_{E^*}e_i = \xi_i$ and $\iota_E e^{*i} = y^i$.

It is easy to see that every linear 2-contravariant tensor Λ on E^* is of the form

$$\Lambda = c^k_{ij}\xi_k\partial_{\xi_i} \otimes \partial_{\xi_j} + \delta^a_i\partial_{\xi_i} \otimes \partial_{x^a} - \sigma^a_i\partial_{x^a} \otimes \partial_{\xi_i}, \tag{2.4}$$

where c^k_{ij}, δ^a_i and σ^a_i are functions of x^a. The correspondence between Λ and a pseudo-Lie algebroid structure is given by the formulae

$$\begin{aligned} [e_i, e_j] = [e_i, e_j]^\Lambda &= c^k_{ij}e_k \\ \alpha^\Lambda_l(e_i) &= \delta^a_i\partial_{x^a} \\ \alpha^\Lambda_r(e_i) &= \sigma^a_i\partial_{x^a} \end{aligned} \tag{2.5}$$

■

Theorem 2.2 *Let $\tau_i\colon E_i \to M,\ i=1,2,$ be vector bundles over M and let $\Psi\colon E_1 \to E_2$ be a vector bundle morphism over the identity on M. Let Λ_i be a linear, 2-contravariant tensor on E_i^*, $i=1,2$. Then*

$$[\Psi(X),\Psi(Y)]^{\Lambda_2} = \Psi([X,Y]^{\Lambda_1})$$

if and only if Λ_2 and Λ_1 are Ψ^*-related, where $\Psi^*\colon E_2^* \to E_1^*$ is the dual morphism.

Proof. The equality $[\Psi(X),\Psi(Y)]^{\Lambda_2} = \Psi([X,Y]^{\Lambda_1})$ is equivalent to the equality

$$\{(\iota_{E_1^*}X)\circ\Psi^*,(\iota_{E_1^*}Y)\circ\Psi^*\}_{\Lambda_2} = \{\iota_{E_2^*}\Psi(X),\iota_{E_2^*}\Psi(Y)\}_{\Lambda_2} = \{\iota_{E_1^*}X,\iota_{E_1^*}Y\}_{\Lambda_1}\circ\Psi^*. \qquad (2.6)$$

Since the exterior derivatives of functions $\iota_{E_1^*}X$ generate $\mathsf{T}^*E_1^*$ over an open-dense subset (E_1^* minus the zero section), the equality 2.6 holds if and only if the tensors Λ_1,Λ_2 are Ψ^*-related. ∎

To the end of this section we assume that Λ is skew-symmetric, i.e., we consider pre-Lie algebroid structures only.

In this case, the bracket $[\,,\,]^\Lambda$, related to Λ, defined on sections of τ can be extended in a standard way (cf. [3, 4]) to the graded bracket on $\Phi(\tau)$. We refer to this bracket as the Schouten-Nijenhuis bracket and we denote it also by $[\,,\,]^\Lambda$.

Moreover, we can define the 'exterior derivative' d^Λ on $\Phi(\pi)$ and the Lie derivative $\pounds^\Lambda_X\colon\Phi(\pi)\to\Phi(\pi)$ along a section $X\in\Gamma(M,E)$. Also the Nijenhuis-Richardson bracket and the Frölicher-Nijenhuis bracket can be defined on $\Phi_1(\pi)=\oplus_{n\in\mathbb{Z}}\Phi_1^n(\pi)$, where $\Phi_1^n(\pi)=\Gamma(M,E\otimes\wedge^nE^*)$. The definitions of these objects are analogous to the definitions in the classical case (cf. [4]).

The bracket $[\,,\,]^\Lambda$ is a Lie bracket (or, equivalently, $(\mathrm{d}^\Lambda)^2=0$) if and only if Λ defines a Lie algebroid structure, i.e., if and only if Λ is a Poisson tensor. In this case, all classical formulae of differential geometry, like $\pounds^\Lambda_X\circ\mathrm{i}_Y-\mathrm{i}_Y\,\pounds^\Lambda_X=\mathrm{i}_{[X,Y]^\Lambda}$ etc., remain valid. We should also mention the vertical tangent lift

$$\mathrm{v}_\tau\colon\Gamma(M,\otimes^k_ME)\to\Gamma(E,\otimes^k_E\mathsf{T}E)$$

given, in local coordinates, by

$$\mathrm{v}_\tau(f(x)e_{i_1}\otimes\cdots\otimes e_{i_k}) = f(x)\partial_{y^{i_1}}\otimes\cdots\otimes\partial_{y^{i_k}}.$$

In particular, $\mathrm{v}_\tau(X\otimes Y)=\mathrm{v}_\tau(X)\otimes\mathrm{v}_\tau(Y)$ ([4]). In the case of the tangent bundle, $E=\mathsf{T}M$, the vertical lift was denoted by v_T in [3]. An analog of the complete tangent lift d_T, studied for the tangent bundle in [3], can be defined as follows.

Theorem 2.3 *Let Λ be a linear bivector field on E^*, which defines a pre-Lie algebroid structure on a vector bundle $\tau\colon E\to M$. Then, there exists a unique v_T-derivation of order 0*

$$\mathrm{d}^\Lambda_\mathsf{T}\colon\ \otimes(\tau)\to\otimes(\tau_E),$$

which satisfies

$$\mathrm{d}^\Lambda_\mathsf{T}(f)=\iota_E\,\mathrm{d}^\Lambda f\quad\textit{for } f\in C^\infty(M), \qquad (2.7)$$

and

$$\iota_{\mathsf{T}^*E}(\mathrm{d}^\Lambda_\mathsf{T}X)\circ\mathcal{R}=\iota_{\mathsf{T}^*E^*}([\Lambda,\iota_{E^*}X])\quad\textit{for } X\in\Phi^1(\tau), \qquad (2.8)$$

where $[\,,\,]$ is the Schouten bracket of multivector fields on E and $\mathcal{R}\colon\mathsf{T}^*E^*\to\mathsf{T}^*E$ is the canonical isomorphism of double vector bundles(see [9, 10]). Moreover, $\mathrm{d}^\Lambda_\mathsf{T}$ is a homomorphism of the Schouten-Nijenhuis brackets:

$$\mathrm{d}^\Lambda_\mathsf{T}([X,Y]^\Lambda)=[\mathrm{d}^\Lambda_\mathsf{T}X,\mathrm{d}^\Lambda_\mathsf{T}Y], \qquad (2.9)$$

if and only if Λ is a Poisson tensor.

SKETCH OF THE PROOF. Let us take $X \in \Phi^1(\tau)$. The hamiltonian vector field $\mathcal{G}^\Lambda(X) = -[\Lambda, \iota_{E^*}X]$ is linear with respect to the tangent vector bundle structure $\mathsf{T}\tau\colon \mathsf{T}E^* \to \mathsf{T}M$ ([10]). Hence, the function $\iota_{\mathsf{T}^*E^*}[\Lambda, \iota_{E^*}X]$ is linear with respect to both vector bundle structures on T^*E^*: over E and over E^*. It follows that there exists a unique (linear) vector field $\mathrm{d}^\Lambda_{\mathsf{T}}X$ on E, such that $\iota_{\mathsf{T}^*E^*}\mathcal{G}^\Lambda(X) = -(\iota_{\mathsf{T}^*E}\,\mathrm{d}^\Lambda_{\mathsf{T}}X) \circ \mathcal{R}$. We have the formula

$$\mathrm{d}^\Lambda_{\mathsf{T}}(fX) = \mathrm{d}^\Lambda_{\mathsf{T}}(f)\,\mathrm{v}_\tau(X) + \mathrm{v}_\tau(f)\,\mathrm{d}^\Lambda_{\mathsf{T}}(X)$$

and, consequently, we can extend $\mathrm{d}^\Lambda_{\mathsf{T}}$ to a v_τ-derivation on $\otimes(\tau)$. Finally, since $\mathcal{R}$ is an anti-Poisson isomorphism, $\mathrm{d}^\Lambda_{\mathsf{T}}$ is a homomorphism of Schouten-Nijenhuis bracket if and only if

$$[\mathcal{G}^\Lambda(X), \mathcal{G}^\Lambda(Y)] = \mathcal{G}^\Lambda([X,Y]^\Lambda)$$

for all $X, Y \in \Phi^1(\tau)$, or, equivalently, if and only if Λ is a Poisson tensor. ■

Remark. Let us define a mapping

$$\mathcal{J}_E\colon \Phi^n_1(\tau) \to \Phi^n(\tau_E)$$

by

$$\mathcal{J}_E(\mu \otimes X) = -\iota_E(\mu)\cdot \mathrm{v}_\tau(X).$$

It has been shown in [4] that $\mathcal{J}_E$ is a homomorphism of the Nijenhuis-Richardson bracket into the Schouten bracket. We have also a mapping

$$\mathcal{G}^\Lambda\colon \Phi^n_1(\pi) \to \Phi^n(\tau_{E^*})\colon K \mapsto \mathcal{G}^\Lambda(K) = [\Lambda, \mathcal{J}_{E^*}(K)],$$

which is, in the case of a Lie algebroid structure, a homomorphism of the Frölicher-Nijenhuis bracket $[\,,\,]^\Lambda_{F-N}$, associated to Λ, into the Schouten bracket ([4]). The bracket $[\,,\,]^\Lambda_{F-N}$ is given by the formula

$$\begin{aligned}[\mu\otimes X, \nu\otimes Y]^\Lambda = \mu\wedge\nu\otimes[X,Y]^\Lambda + \mu\wedge\pounds^\Lambda_X\nu\otimes Y - \pounds^\Lambda_Y\mu\wedge\nu\otimes X \\ + (-1)^\mu(\mathrm{d}^\Lambda\mu\wedge \mathrm{i}_X\,\nu\otimes Y + \mathrm{i}_Y\,\mu\wedge\mathrm{d}^\Lambda\nu\otimes X).\end{aligned} \tag{2.10}$$

In local coordinates

$$\Lambda = \frac{1}{2}c^k_{ij}\xi_k\partial_{\xi_i}\wedge\partial_{\xi_j} + \delta^a_i\partial_{\xi_i}\wedge\partial_{x^a}. \tag{2.11}$$

Then,

$$\mathrm{d}^\Lambda_{\mathsf{T}}(f) = \frac{\partial f}{\partial x^a}\delta^a_j y^j \tag{2.12}$$

and

$$\mathrm{d}^\Lambda_{\mathsf{T}}(X^i e_i) = X^i\delta^a_i\partial_{x^a} + (X^i c^k_{ji} + \frac{\partial X^k}{\partial x^a}\delta^a_j)y^j\partial_{y^k}. \tag{2.13}$$

It follows that, for $P = \frac{1}{2}P^{ij}e_i\wedge e_j$, we have

$$\mathrm{d}^\Lambda_{\mathsf{T}}(P) = P^{ij}\delta^a_j\partial_{y^i}\wedge\partial_{x^a} + (P^{kj}c^i_{lk} + \frac{1}{2}\frac{\partial P^{ij}}{\partial x^a}\delta^a_l)y^l\partial_{y^i}\wedge\partial_{y^j}. \tag{2.14}$$

Remark. For an arbitrary pseudo-Lie algebroid, we can define the right and the left complete lifts with the use of the right and the left hamiltonian vector fields instead of $[\Lambda, \iota_E X]$.

The following theorem describes the complete lifts in terms of Lie derivatives and contractions.

Theorem 2.4 *Given a vector bundle $\tau\colon E \to M$ and a linear bivector field Λ on E^*, we have*

(a) $\mathrm{v}_\tau(X)(\iota_E\mu) = \mathrm{v}_\tau(\mathrm{i}_X\,\mu) = \tau^*\langle X, \mu\rangle$,

(b) $\mathrm{d}^\Lambda_{\mathsf{T}}(X)(\iota_E\mu) = \iota_E(\pounds^\Lambda_X\mu)$,

where $X \in \Phi^1(\tau)$ and $\mu \in \Phi^1(\pi)$.

Proof. The part (a) has been proved in [4], Theorem 15 c). The part (b) follows from the following sequence of identities:

$$\begin{aligned}\pi_E^*(\mathrm{d}_{\mathsf{T}}^{\Lambda}(X)(\iota_E\mu))\circ\mathcal{R} &= \{\iota_{\mathsf{T}^*E}(\mathrm{d}_{\mathsf{T}}^{\Lambda}X),\pi_E^*(\iota_E\mu)\}_{\Lambda_E}\circ\mathcal{R}\\ &= \{\iota_{\mathsf{T}^*E^*}(\mathrm{v}_\pi\,\mu),\iota_{\mathsf{T}^*E^*}([\Lambda,\iota_{E^*}X])\}_{\Lambda_{E^*}}\\ &= \iota_{\mathsf{T}^*E^*}[\mathrm{v}_\pi\,\mu,[\Lambda,\iota_{E^*}X]] = \iota_{\mathsf{T}^*E^*}[\mathcal{G}^{\Lambda}(X),\mathrm{v}_\pi(\mu)]\\ &= \iota_{\mathsf{T}^*E^*}(\mathrm{v}_\pi(\pounds_X^{\Lambda}\mu)) = \pi_E^*(\iota_E(\pounds_X^{\Lambda}\mu))\circ\mathcal{R},\end{aligned}$$

where we used the equalities $[\mathcal{G}^{\Lambda}(X),\mathrm{v}_\pi\,\mu]=\mathrm{v}_\pi(\pounds_X^{\Lambda}\mu)$ (see [4], Theorem 15 e)) and $\iota_{\mathsf{T}^*E^*}\,\mathrm{v}_\pi(\mu)=\pi_E^*(\iota_E\mu)\circ\mathcal{R}$. ■

Theorem 2.5 *If $P\in\Phi^2(\tau)$, then $\mathrm{d}_{\mathsf{T}}^{\Lambda}(P)$ defines a pre-Lie algebroid structure on E^* with the bracket*

$$[\mu,\nu]^{\mathrm{d}_{\mathsf{T}}^{\Lambda}(P)} = \pounds_{P_\mu}^{\Lambda}\nu - \pounds_{P_\nu}^{\Lambda}\mu - \mathrm{d}^{\Lambda}P(\mu,\nu), \tag{2.15}$$

where $P_\mu=\mathrm{i}_\mu P$, and the anchor is given by

$$\alpha^{\mathrm{d}_{\mathsf{T}}^{\Lambda}(P)}(\mu)=\alpha^{\Lambda}(P_\mu).$$

Proof. It is sufficient to consider $P=X\wedge Y$, $X,Y\in\Phi^1(\tau)$. Let us denote $\mathrm{d}_{\mathsf{T}}^{\Lambda}(X)$ and $\mathrm{d}_{\mathsf{T}}^{\Lambda}(Y)$ by $\dot X$ and $\dot Y$, $\mathrm{v}_\tau(X)$ and $\mathrm{v}_\tau(Y)$ by $\bar X$ and $\bar Y$, $\pounds^{\Lambda}$ and d^{Λ} by $\pounds$ and d. Then, we have

$$\begin{aligned}\{\iota_E\mu,\iota_E\nu\}_{\mathrm{d}_{\mathsf{T}}^{\Lambda}(P)} &= \dot X(\iota_E\mu)\bar Y(\iota_E\nu)-\dot X(\iota_E\nu)\bar Y(\iota_E\mu)-\bar X(\iota_E\mu)\dot Y(\iota_E\nu)+\bar X(\iota_E\nu)\dot Y(\iota_E\mu)\\ &= \iota_E\left(\langle Y,\nu\rangle\pounds_X(\mu)-\langle Y,\mu\rangle\pounds_X(\nu)-\langle X,\nu\rangle\pounds_Y(\mu)+\langle X,\mu\rangle\pounds_X(\nu)\right)\\ &= \iota_E\left(\pounds_{P_\mu}\nu-\pounds_{P_\nu}\mu-\mathrm{d}(\langle X,\mu\rangle\langle Y,\nu\rangle-\langle X,\nu\rangle\langle Y,\mu\rangle)\right),\end{aligned}$$

where we used Theorem 2.4. Now, we have

$$[\mu,f\nu]^{\mathrm{d}_{\mathsf{T}}^{\Lambda}(P)}=\pounds_{P_\mu}(f\nu)-\pounds_{P_{f\nu}}\mu-\mathrm{d}P(\mu,f\nu)=f[\mu,\nu]^{\mathrm{d}_{\mathsf{T}}^{\Lambda}(P)}+(\pounds_{P_\mu}f)\nu,$$

so that $\alpha^{\mathrm{d}_{\mathsf{T}}^{\Lambda}(P)}(\mu)=\pounds_{P_\mu}(f)=\alpha^{\Lambda}(P_\mu)(f)$. ■

In the case of the canonical Lie algebroid on the tangent bundle $\tau_M\colon \mathsf{T}M\to M$, associated with the canonical Poisson tensor Λ_M on T^*M, our definition of $\mathrm{d}_{\mathsf{T}}^{\Lambda}(P)$ gives the standard tangent complete lift d_{T}. Moreover, the bracket 2.15 of 1-forms is the bracket introduced independently in [2, 7, 6] and corresponding to the lift $\mathrm{d}_{\mathsf{T}}P$ (cf. [1, 3]).

Example. Let us consider a Lie algebroid over a point, i.e., a real Lie algebra $\mathfrak{g}$ with a basis $e_1,\dots,e_m$, and its dual space $\mathfrak{g}^*$ with the dual basis $e^{*1},\dots,e^{*m}$. We have also the corresponding (linear) coordinate system $\xi_1,\dots,\xi_m$ on $\mathfrak{g}^*$ and the coordinate system $y^1,\dots,y^m$ on $\mathfrak{g}$. The linear Poisson structure Λ on $\mathfrak{g}^*$, associated with the Lie bracket $[\,,]^{\Lambda}$ on $\mathfrak{g}$, is the well-known Kostant-Kirillov-Souriau tensor

$$\Lambda=\frac{1}{2}c_{ij}^k\xi_k\partial_{\xi_i}\wedge\partial_{\xi_j}.$$

Here c_{ij}^k are the structure constants with respect to the chosen basis. The exterior derivative $\mathrm{d}^{\Lambda}\colon\wedge\mathfrak{g}^*\to\wedge\mathfrak{g}^*$ is the dual mapping to the Lie bracket:

$$\mathrm{d}^{\Lambda}(\mu)(X,Y)=\langle\mu,[Y,X]^{\Lambda}\rangle,$$

i.e., d^{Λ} is the Chevalley cohomology operator. For $x\in\Phi^1(\tau)=\mathfrak{g}$, the tangent complete lift $\mathrm{d}_{\mathsf{T}}^{\Lambda}(x)$ is the fundamental vector field of the adjoint representation, corresponding to x:

$$\mathrm{d}_{\mathsf{T}}^{\Lambda}(e_i)=c_{ji}^k y^j\partial_{y^k}.$$

3. NIJENHUIS TENSORS AND POISSON-NIJENHUIS STRUCTURES FOR LIE ALGEBROIDS

Let a vector bundle $\tau\colon E \to M$ be given a pseudo-Lie algebroid structure, associated with a tensor field Λ on E^*, and let $\widetilde{N}\colon E \to E$ be a vector bundle morphism over the identity. We can represent $\widetilde{N}$, as well as its dual $\widetilde{N}^*$, by a tensor field $N \in \Phi^1_1(\pi)$. This tensor field defines operations in $\Phi^1(\tau)$ and $\Phi^1(\pi)$, which we denote by the same symbol i_N. If $N = X_i \otimes \mu^i,\ X_i \in \Phi^1(\tau),\ \mu^i \in \Phi^1(\pi)$, the operation i_N is given by the formulae

$$\mathrm{i}_N X = \langle X, \mu^i\rangle X_i \quad \text{and} \quad \mathrm{i}_N \mu = \langle X_i, \mu\rangle \mu^i,$$

where $X \in \Phi^1(\tau),\ \mu \in \Phi^1(\pi)$. In the notation of [6], $\mathrm{i}_N X = NX$ and $\mathrm{i}_N \mu = {}^t N\mu$. It is obvious that we can extend i_N to a derivation of the tensor algebra, putting

$$\mathrm{i}_N(A \otimes B) = (\mathrm{i}_N A) \otimes B + A \otimes (\mathrm{i}_N B). \tag{3.1}$$

Using N, we can deform the bracket $[\,,\,]^\Lambda$ to a bracket $[\,,\,]^\Lambda_N$ on $\Phi^1(\tau)$ by the formula

$$[X,Y]^\Lambda_N = [NX,Y]^\Lambda + [X,NY]^\Lambda - N[X,Y]^\Lambda. \tag{3.2}$$

Theorem 3.1 *The deformed bracket 3.2 defines on E a pseudo-Lie algebroid structure, with the anchors $(\alpha^\Lambda_N)_l = \alpha^\Lambda_l \circ \widetilde{N}$ and $(\alpha^\Lambda_N)_r = \alpha^\Lambda_r \circ \widetilde{N}$. The associated tensor field is given by*

$$\Lambda_N = \pounds_{\mathcal{J}_{E^*}(N)}\Lambda,$$

where $\pounds_{\mathcal{J}_{E^*}(N)}$ is the standard Lie derivative along the vector field $\mathcal{J}_{E^*}(N)$.

If Λ is skew-symmetric, then Λ_N is also skew-symmetric and the Schouten-Nijenhuis bracket, induced by Λ_N, can be written in the form, similar to 3.2,

$$[X,Y]^{\Lambda_N} = [X,Y]^\Lambda_N = [\mathrm{i}_N X, Y]^\Lambda + [X, \mathrm{i}_N Y]^\Lambda - \mathrm{i}_N([X,Y]^\Lambda) \tag{3.3}$$

for $X, Y \in \Phi(\tau)$. Moreover,

$$\mathrm{d}^{\Lambda_N} = \mathrm{i}_N \circ \mathrm{d}^\Lambda - \mathrm{d}^\Lambda \circ \mathrm{i}_N\,. \tag{3.4}$$

The proof is based on the following Lemma.

Lemma 3.1 *For $X \in \Phi^1(\tau)$, we have*

$$\iota_{E^*}(NX) = -\pounds_{\mathcal{J}_{E^*}(N)}(\iota_{E^*}) \tag{3.5}$$

and

$$\pounds_{\mathcal{J}_{E^*}(N)}\,\mathrm{v}_\pi(\mu) = \mathrm{v}_\pi(\mathrm{i}_N \mu) \tag{3.6}$$

for $X \in \Phi^1(\tau),\ \mu \in \Phi^1(\pi)$.

Proof. Let $N = X_i \otimes \mu^i$, $X_i \in \Phi^1(\tau)$ and $\mu^i \in \Phi^1(\pi)$. We have

$$\begin{aligned}\iota_{E^*}(NX) &= \iota_{E^*}(\langle X, \mu^i\rangle X_i) = \pi^*(\langle X, \mu^i\rangle)\iota_{E^*}(X_i)\\ &= \mathrm{v}_\pi(\langle X, \mu^i\rangle)\iota_{E^*}(X_i).\end{aligned}$$

On the other hand,

$$\begin{aligned}-\pounds_{\mathcal{J}_{E^*}(N)}(\iota_{E^*}X) &= (\iota_{E^*}(X_i)\,\mathrm{v}_\pi(\mu^i))(\iota_{E^*}X) = \iota_{E^*}(X_i)\,\mathrm{v}_\pi(\mu^i)(\iota_{E^*}X)\\ &= \iota_{E^*}(X_i)\,\mathrm{v}_\pi(\langle X, \mu^i\rangle),\end{aligned}$$

according to Theorem 15 c) in [4].

Similarly,

$$\begin{aligned}[\mathcal{J}_{E^*}(N), \mathrm{v}_\pi(\mu)] &= [\iota_{E^*}(X_i)\,\mathrm{v}_\pi(\mu_i), \mathrm{v}_\pi(\mu)]\\ &= -\,\mathrm{v}_\pi(\mu^i) \wedge [\iota_{E^*}(X_i), \mathrm{v}_\pi(\mu)],\end{aligned}$$

since the vertical vector fields commute. Following Theorem 15 c) in [4], we get

$$[\iota_{E^*}(X_i), \mathrm{v}_\pi(\mu)] = -\,\mathrm{v}_\pi(\mathrm{i}_{X_i}\,\mu)$$

and, consequently,

$$[\mathcal{J}_{E^*}(N), \mathrm{v}_\pi(\mu)] = \mathrm{v}_\pi(\mu^i \wedge \mathrm{i}_{X_i}\,\mu) = \mathrm{v}_\pi(\mathrm{i}_N\,\mu).$$

$\blacksquare$

Proof of Theorem 3.1. Using Lemma and properties of the Lie derivative, we get

$$\begin{aligned}\iota_{E^*}([X,Y]^\Lambda_N) &= \iota_{E^*}\left([NX,Y]^\Lambda + [X,NY]^\Lambda - N[X,Y]^\Lambda\right)\\ &= -\{\pounds_{\mathcal{J}_{E^*}(N)}(\iota_{E^*}X), \iota_{E^*}Y\}_\Lambda - \{\iota_{E^*}X, \pounds_{\mathcal{J}_{E^*}(N)}(\iota_{E^*}Y)\}_\Lambda +\\ &\quad + \pounds_{\mathcal{J}_{E^*}(N)}\{\iota_{E^*}X, \iota_{E^*}Y\}_\Lambda\\ &= \{\iota_{E^*}X, \iota_{E^*}Y\}_{\pounds_{\mathcal{J}_{E^*}(N)}\Lambda}.\end{aligned}$$

The general form 3.3 of the corresponding Schouten bracket follows inductively from the Leibniz rule for the Schouten bracket $[\,,\,]^\Lambda$ and from 3.1.

In order to prove 3.4, we, again, use Lemma and [4] Theorem 15 d):

$$\begin{aligned}\mathrm{v}_\pi(\mathrm{d}^{\Lambda_N}\mu) &= [\Lambda_N, \mathrm{v}_\pi\,\mu] = [[\mathcal{J}_{E^*}N, \Lambda], \mathrm{v}_\pi\,\mu]\\ &= [\mathcal{J}_{E^*}N, [\Lambda, \mathrm{v}_\pi\,\mu]] - [\Lambda, [\mathcal{J}_{E^*}N, \mathrm{v}_\pi\,\mu]]\\ &= [\mathcal{J}_{E^*}N, \mathrm{v}_\pi(\mathrm{d}_\Lambda\mu)] - [\Lambda, \mathrm{v}_\pi(\mathrm{i}_N\,\mu)] = \mathrm{v}_\pi(\mathrm{i}_N\,\mathrm{d}^\Lambda\mu - \mathrm{d}^\Lambda\,\mathrm{i}_N\,\mu).\end{aligned}$$

$\blacksquare$

In local coordinates, we have

$$\begin{aligned}N &= N^i_j e_i \otimes e^{*j},\\ \Lambda &= c^k_{ij}\xi_k\partial_{\xi_i} \otimes \partial_{\xi_j} + \delta^a_i\partial_{\xi_i} \otimes \partial_{x^a} - \sigma^a_i\partial_{x^a} \otimes \partial_{\xi_i},\\ \mathcal{J}_{E^*}N &= N^i_k\xi_i\partial_{\xi_k},\end{aligned}$$

and

$$\begin{aligned}\Lambda_N = &\left(c^k_{lj}N^l_i + c^k_{il}N^l_j - c^l_{ij}N^k_l + \delta^a_i\frac{\partial N^k_j}{\partial x^a} - \sigma^a_j\frac{\partial N^k_i}{\partial x^a}\right)\xi_k\partial_{\xi_i} \otimes \partial_{\xi_j}\\ &+ N^l_i\delta^a_l\partial_{\xi_i} \otimes \partial_{x^a} - N^l_i\sigma^a_l\partial_{x^a} \otimes \partial_{\xi_i}. \quad (3.7)\end{aligned}$$

Theorem 3.2 *For $X \in \otimes(\tau)$ and skew-symmetric Λ, we have*

$$\mathrm{d}^{\Lambda_N}_{\mathsf{T}}(X) = \mathrm{d}^\Lambda_{\mathsf{T}}(\mathrm{i}_N\,X) - \pounds_{\mathcal{J}_E(N)}\,\mathrm{d}^\Lambda_{\mathsf{T}}(X).$$

Proof. Since $\mathrm{d}^{\Lambda_N}_{\mathsf{T}}$ and $\mathrm{d}^\Lambda_{\mathsf{T}}$ are v_τ-derivations of order 0 on $\otimes(\tau)$ and $\pounds_{\mathcal{J}_E(N)}\,\mathrm{v}_\tau(X) = 0$ ($\mathcal{J}_E(N)$ is vertical), it is enough to consider the case $X \in \Phi^1(\tau)$. For such X

$$\begin{aligned}\iota_{\mathsf{T}^*E}\left(\mathrm{d}^{\Lambda_N}_{\mathsf{T}}(X)\right) \circ \mathcal{R} &= \iota_{\mathsf{T}^*E^*}[\Lambda_N, \iota_{E^*}X] = \iota_{\mathsf{T}^*E^*}[\pounds_{\mathcal{J}_{E^*}(N)}\Lambda, \iota_{E^*}X]\\ &= \iota_{\mathsf{T}^*E^*}\left(\pounds_{\mathcal{J}_{E^*}(N)}[\Lambda, \iota_{E^*}X\right) - \iota_{\mathsf{T}^*E^*}[\Lambda, \pounds_{\mathcal{J}_{E^*}(N)}(\iota_{E^*}X)].\end{aligned}$$

Since $\pounds_{\mathcal{J}_{E^*}(N)} = -\iota_{E^*}(NX)$ (3.5), then

$$-\iota_{\mathsf{T}^*E^*}[\Lambda, \pounds_{\mathcal{J}_{E^*}(N)}(\iota_{E^*}X)] = \iota_{\mathsf{T}^*E^*}[\Lambda, \iota_{E^*}(NX)] = \iota_{E^*}\left(\mathrm{d}^\Lambda_{\mathsf{T}}(NX)\right) \circ \mathcal{R}.$$

On the other hand,

$$\begin{aligned}
\iota_{\mathsf{T}^*E^*}\left(\pounds_{\mathcal{J}_{E^*}(N)}[\Lambda,\iota_{E^*}X\right) &= \{\iota_{\mathsf{T}^*E^*}(\mathcal{J}_{E^*}(N)),\iota_{\mathsf{T}^*E^*}[\Lambda,\iota_{E^*}X]\}_{\Lambda_{E^*}}\\
&= \{\iota_{\mathsf{T}^*E}(\mathcal{J}_E N)\circ\mathcal{R},\iota_{\mathsf{T}^*E}(\mathrm{d}^\Lambda_{\mathsf{T}}(X))\circ\mathcal{R}\}_{\Lambda_{E^*}}\\
&= -\{\iota_{\mathsf{T}^*E}(\mathcal{J}_E N),\iota_{\mathsf{T}^*E}(\mathrm{d}^\Lambda_{\mathsf{T}}(X))\}_{\Lambda_E}\circ\mathcal{R}\\
&= -\left(\iota_{\mathsf{T}^*E}[\mathcal{J}_E N,\mathrm{d}^\Lambda_{\mathsf{T}}(X)]\right)\circ\mathcal{R}\\
&= -\iota_{\mathsf{T}^*E}\left(\pounds_{\mathcal{J}_E(N)}\,\mathrm{d}^\Lambda_{\mathsf{T}}(X)\right),
\end{aligned}$$

where we used the equality $\iota_{\mathsf{T}^*E}(\mathcal{J}_E N)=\iota_{\mathsf{T}^*E^*}(\mathcal{J}_{E^*}(N))$. Since $\mathcal{R}$ is an isomorphism and ι_{T^*E} is injective, the theorem follows. ■

In local coordinates, for Λ as in 2.11, we have

$$\mathrm{d}^{\Lambda_N}_{\mathsf{T}}(X^i e_i)=X^i N^k_i\delta^a_k\partial_{x^a}+ \\
+\left(X^i(N^k_j c^n_{ki}+N^k_i c^n_{jk}-N^n_k c^k_{ji}+\delta^a_j\frac{\partial N^n_i}{\partial x_a}-\delta^a_i\frac{\partial N^n_j}{\partial x_a})+\frac{\partial X^n}{\partial x^a}N^k_j\delta^a_k\right)y_j\partial_{y^n}. \quad (3.8)$$

Remark. If we treat the Schouten brackets $B^\Lambda=[\,,\,]^\Lambda$ and $B^\Lambda_N=[\,,\,]^\Lambda_N$ as bilinear operators on $\Phi(\tau)$, then fomula 3.2 means

$$B^\Lambda_N=[\mathrm{i}_N,B^\Lambda]_{N-R}, \quad (3.9)$$

where $[\,,\,]_{N-R}$ is the Nijenhuis-Richardson bracket of multilinear graded operators of a graded space in the sense of [8]. Similarly, 3.4 means that

$$\mathrm{d}^{\Lambda_N}_{\mathsf{T}}=[\mathrm{i}_N,\mathrm{d}^\Lambda_{\mathsf{T}}]_{N-R}. \quad (3.10)$$

This interpretation will be used later, together with the Jacobi identity for the $[\,,\,]_{N-R}$.

Definition 3.1 *A tensor $N\in\Gamma(M,E\otimes E^*)$ is called a* Nijenhuis tensor *for Λ (or, for a Lie algebroid structure defined by Λ), if the* Nijenhuis torsion

$$T^\Lambda_N(X,Y)=N[X,Y]^\Lambda_N-[NX,NY]^\Lambda \quad (3.11)$$

vanishes for all $X,Y\in\Gamma(E)$.

The classical version of the following is well known (cf. [6]).

Theorem 3.3

(a) *N is a Nijenhuis tensor for Λ if and only if Λ and $\Lambda_N=\pounds_{\mathcal{J}_{E^*}(N)}\Lambda$ are $\widetilde{N}^*$-related.*

(b) *The Nijenhuis torsion corresponds to the Frölicher-Nijenhuis bracket:*

$$T^\Lambda_N(X,Y)=\frac{1}{2}[N,N]^\Lambda_{F-N}(X,Y).$$

(c)

$$[[B^\Lambda,\mathrm{i}_N]_{R-N},\mathrm{i}_N]_{N-R}=2T^\Lambda_N+[B^\Lambda,\mathrm{i}_{N^2}]_{N-R},$$

where $(X_i\otimes\mu^i)^2=\langle X_i,\mu^j\rangle\mu^i\otimes X_j$.

(d) *If N is a Nijenhuis tensor, then Λ_N is a Poisson tensor.*

Proof.

(a) Since $\Lambda_N=\pounds_{\mathcal{J}_{E^*}(N)}\Lambda$ induces the deformed bracket $B^\Lambda_N=[\,,\,]^\Lambda_N$, this part follows from Theorem 2.2.

(b) Let $N=X_i\otimes\mu^i$, then

$$[NX,NY]^\Lambda=\langle X,\mu^i\rangle\langle Y,\mu^j\rangle[X_i,X_j]^\Lambda+\langle X,\mu^i\rangle\pounds^\Lambda_{X_i}(\langle Y,\mu^j\rangle)X_j-\langle Y,\mu^j\rangle\pounds^\Lambda_{X_j}(\langle X,\mu^i\rangle)X_i$$

and

$$
\begin{aligned}
N[X,Y]^\Lambda_N = N\big(&\langle X,\mu^i\rangle[X_i,Y]^\Lambda - \pounds^\Lambda_Y(\langle X,\mu^i\rangle)X_i + \langle Y,\mu^j\rangle[X,X_j] \\
&+ \pounds^\Lambda_X(\langle Y,\mu^j\rangle)X_j - \langle [X,Y]^\Lambda,\mu^i\rangle X_i\big) \\
= {}&\langle X,\mu^i\rangle\langle [X_i,Y]^\Lambda,\mu^j\rangle X_j - \pounds^\Lambda_Y(\langle X,\mu^i\rangle)\langle X_i,\mu^j\rangle X_j \\
&+ \langle Y,\mu^j\rangle\langle [X,X_j]^\Lambda,\mu^i\rangle X_i + \pounds^\Lambda_X(\langle Y,\mu^j\rangle)\langle X_j,\mu^i\rangle X_i - \langle [X,Y]^\Lambda,\mu^i\rangle\langle X_i,\mu^j\rangle X_j. \quad (3.12)
\end{aligned}
$$

Hence, using properties of Lie derivatives, we get

$$
\begin{aligned}
T^\Lambda_N(X,Y) &= \langle X,\mu^i\rangle\langle Y,\mu^j\rangle[X_i,X_j]^\Lambda + \langle X,\mu^i\rangle\langle Y,\pounds^\Lambda_{X_i}\mu^j\rangle X_j \\
&\quad - \langle Y,\mu^j\rangle\langle X,\pounds^\Lambda_{X_j}\mu^i\rangle X_i + \mathrm{d}\mu^i(X,Y)\langle X_i,\mu^j\rangle X_j \\
&= \left(\frac{1}{2}\mu^i\wedge\mu^j\otimes[X_i,X_j]^\Lambda + \mu_i\wedge\pounds^\Lambda_{X_i}\mu^j\otimes X_j + \mathrm{d}^\Lambda\mu^i\wedge \mathrm{i}_{X_i}\mu^j\otimes X_j\right)(X,Y) \\
&= \frac{1}{2}[N,N]^\Lambda_{F-N}(X,Y).
\end{aligned}
$$

(c)

$$
\begin{aligned}
&[[B^\Lambda,\mathrm{i}_N]_{R-N},\mathrm{i}_N]_{R-N}(X,Y) = \\
&\quad = ([N^2X,Y]^\Lambda + [NX,NY]^\Lambda - N[NX,Y]^\Lambda) + ([NX,NY]^\Lambda + [X,N^2Y]^\Lambda - N[X,NY]^\Lambda) \\
&\qquad - (N[NX,Y]^\Lambda + N[X,NY]^\Lambda - N^2[X,Y]^\Lambda) \\
&\quad = 2([NX,NY]^\Lambda - N([NX,Y]^\Lambda + [X,NY]^\Lambda - N[X,Y]^\Lambda)) \\
&\qquad + [N^2X,Y]^\Lambda + [X,N^2Y]^\Lambda - N^2[X,Y]^\Lambda \\
&\quad = 2T^\Lambda_N(X,Y) + [B^\Lambda,\mathrm{i}_{N^2}]_{N-R}(X,Y). \quad (3.13)
\end{aligned}
$$

(d) The Schouten bracket induced by Λ_N is given by $[\mathrm{i}_N,B^\Lambda]_{N-R}$ and it is known from general theory [8], that it defines a graded Lie algebra structure if and only if its Nijenhuis-Richardson square vanishes. Using the graded Jacobi identity for $[\,,\,]_{N-R}$, we get

$$
\begin{aligned}
[[\mathrm{i}_N,B^\Lambda]_{N-R},[\mathrm{i}_N,B^\Lambda]_{N-R}]_{N-R} &= [[[\mathrm{i}_N,B^\Lambda]_{N-R},\mathrm{i}_N]_{N-R},B^\Lambda]_{N-R} \\
&= -2[T^\Lambda_N,B^\Lambda]_{N-R} + [[\mathrm{i}_{N^2},B^\Lambda]_{N-R},B^\Lambda]_{N-R} \\
&= 0,
\end{aligned}
$$

since $T^\Lambda_N = 0$ and $[B^\Lambda,B^\Lambda]_{N-R} = 0$ ($[\,,\,]^\Lambda$ is a Lie bracket) implies that $\left(\mathrm{ad}^{N-R}_{B^\Lambda}\right)^2 = 0$. ■

The following theorem is, essentially, due to Mackenzie and Xu ([9]).

Theorem 3.4 *Let Λ be a Poisson tensor on E^* and let $P\in\Phi^2(\tau)$. Then*

(a) $\mathrm{d}^\Lambda_{\mathsf{T}}(P)$ *induces a pre-Lie algebroid structure on E^*, with the bracket and the anchor described in Theorem 2.5. The exterior derivative, induced by $\mathrm{d}^\Lambda_{\mathsf{T}}(P)$, is given by*

$$\mathrm{d}^{\mathrm{d}^\Lambda_{\mathsf{T}}(P)}(X) = [P,X]^\Lambda. \quad (3.14)$$

Moreover,

$$\frac{1}{2}[P,P]^\Lambda(\mu,\nu,\gamma) = \langle \widetilde{P}([\mu,\nu]^{\mathrm{d}^\Lambda_{\mathsf{T}}(P)}) - [P_\mu,P_\nu]^\Lambda,\gamma\rangle \quad (3.15)$$

for all $\mu,\nu,\gamma\in\Phi^1(\pi)$ and P is a Poisson tensor for Λ (i.e., $[P,P]^\Lambda = 0$) if and only if Λ and $\mathrm{d}^\Lambda_{\mathsf{T}}(P)$ are $-\widetilde{P}$-related, where $\widetilde{P}(\mu) = P_\mu = \mathrm{i}_\mu P$.

(b) *if P is, in addition, a Poisson tensor for Λ, then $\mathrm{d}^\Lambda_{\mathsf{T}}(P)$ is a Poisson tensor and Poisson tensors $\Lambda,\mathrm{d}^\Lambda_{\mathsf{T}}(P)$ induce a Lie bialgebroid structure on bundles E and E^*, i.e.,*

$$\mathrm{d}^\Lambda\left([\mu,\nu]^{\mathrm{d}^\Lambda_{\mathsf{T}}(P)}\right) = [\mathrm{d}^\Lambda\mu,\nu]^{\mathrm{d}^\Lambda_{\mathsf{T}}(P)} + (-1)^{\mu+1}[\mu,\mathrm{d}^\Lambda\nu]^{\mathrm{d}^\Lambda_{\mathsf{T}}(P)}. \quad (3.16)$$

Proof. The proof of 3.15 is completely analogous to the proof in the classical case (see [6]). The remaining part of (a) follows from Theorem 2.2. Part(b) is proved in [9].

■

Remark. Due to the result of Kosmann-Schwarzbach ([5]), 3.16 is equivalent to

$$[P,[X,Y]^{\Lambda}]^{\Lambda} = [[P,X]^{\Lambda},Y]^{\Lambda} + (-1)^{X}[X,[P,Y]^{\Lambda}]^{\Lambda}, \tag{3.17}$$

which is a special case of the graded Jacobi identity for the bracket $[\,,\,]^{\Lambda}$.

The fact that $\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)$ is a Poisson tensor, if $[P,P]^{\Lambda} = 0$, is a direct consequence of 2.11. The converse to this is not true, in general, as shows the following example.

Example 2. For a Lie algebroid over a point, i.e., for a Lie algebra $\mathfrak{g}$ with the bracket $[\,,\,]^{\Lambda}$, corresponding to a Kirillov-Kostant-Souriau tensor Λ on $\mathfrak{g}^*$, $P \in \wedge^2\mathfrak{g}$ is a Poisson tensor for Λ if and only if P is an r-matrix satisfying the classical Yang-Baxter equation $[P,P]^{\Lambda} = 0$.

On the other hand, $\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)$ is a Poisson tensor if and only if $\mathrm{d}^{\Lambda}_{\mathsf{T}}([P,P]^{\Lambda}) = 0$ which means, that $\mathrm{ad}_{\xi}[P,P]^{\Lambda} = 0$ for all $\xi \in \mathfrak{g}$, i.e., the equation $\mathrm{d}^{\Lambda}_{\mathsf{T}}([P,P]^{\Lambda}) = 0$ is the modified Yang-Baxter equation.

Definition 3.2 *Let $P \in \Phi^2(\tau)$ be a Poisson tensor with respect to a Lie algebroid structure on E, associated to a Poisson tensor Λ on E^*, and let $N \in \Phi^1_1(\tau)$ be a Nijenhuis tensor for Λ. We call the pair (P,N) a* Poisson-Nijenhuis structure *for Λ if the following two conditions are satisfied:*

1. *$NP = PN^*$, where $NP(\mu,\nu) = P(\mu,\mathrm{i}_N\,\nu)$ and $PN^*(\mu,\nu) = P(\mathrm{i}_N\,\mu,\nu)$,*
2. *$\mathrm{d}^{\Lambda_N}_{\mathsf{T}}(P) = \left(\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)\right)_N$.*

Remark. Since $NP + PN^* = \mathrm{i}_N\,P$ and, according to Theorems 3.1 and 3.2,

$$\left(\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)\right)_N = \pounds_{\mathcal{J}_E(N)}\,\mathrm{d}^{\Lambda}_{\mathsf{T}}(P),$$

$$\mathrm{d}^{\Lambda_N}_{\mathsf{T}}(P) = \mathrm{d}^{\Lambda}_{\mathsf{T}}(\mathrm{i}_N\,P) - \pounds_{\mathcal{J}_E(N)}\,\mathrm{d}^{\Lambda}_{\mathsf{T}}(P),$$

the condition (2) can be replaced by
(2') $\pounds_{\mathcal{J}_E(N)}\,\mathrm{d}^{\Lambda}_{\mathsf{T}}(P) = \left(\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)\right)_N = \mathrm{d}^{\Lambda}_{\mathsf{T}}(NP)$.

Theorem 3.5 *If (P,N) is a Poisson-Nijenhuis structure for Λ then NP is a Poisson tensor for Λ and we have the following commutative diagram of Poisson mappings between Poisson manifolds.*

$$\begin{array}{ccc} (E^*,\Lambda) & \xrightarrow{-\widetilde{P}} & (E,\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)) \\ \Big\downarrow \widetilde{N}^* & & \Big\downarrow \widetilde{N} \\ (E^*,\Lambda_N) & \xrightarrow{-\widetilde{P}} & (E,\mathrm{d}^{\Lambda}_{\mathsf{T}}(NP) = \left(\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)\right)_N) \end{array},$$

where $\Lambda_N = \pounds_{\mathcal{J}_{E^*}(N)}\Lambda$ and $\left(\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)\right)_N = \pounds_{\mathcal{J}_E(N)}\,\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)$. Moreover, every structure from the left-hand side of this diagram constitutes a Lie bialgebroid structure with every right-hand side structure.

Proof. The tensors Λ_N and $\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)$ are Poisson. The mappings $-\widetilde{P}\colon (E^*,\Lambda) \to (E,\mathrm{d}^{\Lambda}_{\mathsf{T}}(P))$ and $\widetilde{N}^*\colon (E,\Lambda) \to (E^*,\Lambda_N)$ are Poisson, in view of Theorems 3.3 and 3.4. The assumption $NP = PN^*$ implies that the diagram is commutative. To show that the mapping $-\widetilde{P}\colon (E^*,\Lambda_N) \to (E,\left(\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)\right)_N)$ is Poisson, it is enough to check that, under the assumption $NP = PN^*$, the vector fields $\mathcal{J}_{E^*}(N)$ and $\mathcal{J}_E(N)$ are $-\widetilde{P}$-related. One can do it easily. Since Λ and $\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)$ are $-\widetilde{P}$-related, also $\Lambda_N = [\mathcal{J}_{E^*}(N),\Lambda]$ and $\left(\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)\right)_N = [\mathcal{J}_E(N),\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)]$ are $-\widetilde{P}$-related. Hence, the equality $\left(\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)\right)_N = \mathrm{d}^{\Lambda}_{\mathsf{T}}(NP)$ implies that Λ and $\mathrm{d}^{\Lambda}_{\mathsf{T}}(NP)$ are $-\widetilde{NP}$-related and, according to Theorem 3.4 a), NP is a Poisson tensor for Λ.

The fact that the mapping $\widetilde{N}\colon (E,\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)) \to (E,\left(\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)\right)_N)$ is Poisson follows from the identity

$$\begin{aligned} \langle X, [N,N]^{\mathrm{d}^{\Lambda}_{\mathsf{T}}(P)}_{F-N}(\alpha,\beta)\rangle &= \langle [N,N]^{\Lambda}_{F-N}(X,P_{\beta}),\alpha\rangle \\ &\quad + 2\langle X, C^{\Lambda}(P,N)(\mathrm{i}_N\,\alpha,\beta)\rangle - 2\langle NX, C^{\Lambda}(P,N)(\alpha,\beta)\rangle, \end{aligned} \tag{3.18}$$

where $C^\Lambda(P,N)(\alpha,\beta)=[\alpha,\beta]^{\mathrm{d}^\Lambda_{\mathsf{T}}(NP)}-[\alpha,\beta]_N^{\mathrm{d}^\Lambda_{\mathsf{T}}(P)}$. This is a generalization of an analogous identity in [6], with a completely parallel proof.

The pairs $(\Lambda,\mathrm{d}^\Lambda_{\mathsf{T}}(P))$ and $(\Lambda,\mathrm{d}^\Lambda_{\mathsf{T}}(NP))$ constitute Lie bialgebroids by Theorem 3.4 b), since P and NP are Poisson tensors for Λ.

Similarly, $(\Lambda_N,\mathrm{d}^\Lambda_{\mathsf{T}}(NP))=\mathrm{d}^{\Lambda_N}_{\mathsf{T}}(P)$ constitute a Lie bialgebroid, since P is a Poisson tensor for Λ_N (Λ_N and $\mathrm{d}^{\Lambda_N}_{\mathsf{T}}(P)$ are $-\widetilde{P}$-related).

To show that the pair $(\Lambda_N,\mathrm{d}^\Lambda_{\mathsf{T}}(P))$ forms a Lie bialgebroid, we have to prove that $\mathrm{d}^{\Lambda_N}=\mathrm{d}^\Lambda_N$ is a derivation of the Schouten bracket $B=[\,,\,]^{\mathrm{d}^\Lambda_{\mathsf{T}}(P)}$, i.e., $[\mathrm{d}^\Lambda_N,B]_{N-R}=0$. Since, due to 3.4, $\mathrm{d}^\Lambda_N=[\mathrm{i}_N,\mathrm{d}^\Lambda]_{N-R}$ and $[\mathrm{d}^\Lambda,B]_{N-R}=0$, we get

$$
\begin{aligned}
[\mathrm{d}^\Lambda_N,B]_{N-R}=[[\mathrm{i}_N,\mathrm{d}^\Lambda]_{N-R},B]_{N-R}&=[\mathrm{i}_N,[\mathrm{d}^\Lambda,B]_{N-R}]_{N-R}\\
&+[\mathrm{d}^\Lambda,[\mathrm{i}_N,B]_{N-R}]_{N-R}=[\mathrm{d}^\Lambda,B_{(\mathrm{d}^\Lambda_{\mathsf{T}}(P))_N}]_{N-R}=0, \qquad (3.19)
\end{aligned}
$$

in view of the fact that $[\mathrm{i}_N,B]_{N-R}$ is the bracket associated to $\left(\mathrm{d}^\Lambda_{\mathsf{T}}(P)\right)_N$, for which d^Λ is a derivation. ■

Remark. The above diagram is a dualization of a similar diagram in [6].

In the case of the canonical Lie algebroid on $E=\mathsf{T}M$, the fact that $((\Lambda_M)_N,\mathrm{d}_{\mathsf{T}}P)$ constitutes a Lie bialgebroid is equivalent to the fact that (P,N) is a Poisson-Nijenhuis structure, as it was recently shown in [5]. This is due to the formulae

$$A(f,g)=\langle(NP-PN^*)\,\mathrm{d}^\Lambda g,\mathrm{d}^\Lambda f\rangle, \qquad (3.20)$$

$$A(\mathrm{d}^\Lambda f,g)=C^\Lambda(P,N)(\mathrm{d}^\Lambda f,\mathrm{d}^\Lambda g)+\mathrm{d}^\Lambda A(f,g), \qquad (3.21)$$

where $A=[\mathrm{d}^{\Lambda_N},B_{\mathrm{d}^\Lambda_{\mathsf{T}}(P)}]_{N-R}$, and the fact that A satisfies a Leibniz rule and $\mathrm{d}^\Lambda f$ generate T^*M, for $\Lambda=\Lambda_M$.

In general, this is not true and we can have $((\Lambda)_N,\mathrm{d}_{\mathsf{T}}P)$ being a Lie bialgebroid with (P,N) not being Poisson-Nijenhuis structure for Λ, even if we assume the equality $NP=PN^*$, as shows the following example.

Example. As a Lie algebroid over a single point, let us take a Lie algebra $\mathfrak{g}$ spanned by ξ_1,ξ_2,ξ_3,ξ_4 with the bracket defined by $\Lambda=\xi_3\partial_{\xi_1}\wedge\partial_{\xi_2}$. The tensor $P=\partial_{\xi_2}\wedge\partial_{\xi_4}$ is a Poisson tensor with $\mathrm{d}^\Lambda_{\mathsf{T}}P=y_1\partial_{y_3}\wedge\partial_{y_4}$.

The tensor

$$N=-\xi_1\otimes y_1+\sum_{i=2}^{4}\xi_i\otimes y_i$$

is a Nijenhuis tensor for Λ and $\Lambda_N=-\Lambda$. Moreover, $NP=PN^*=P$, so that $(\Lambda,\mathrm{d}^\Lambda_{\mathsf{T}}(NP))$ constitutes a Lie bialgebroid. In this case, however, $\mathrm{d}^{\Lambda_N}_{\mathsf{T}}P=-\mathrm{d}^\Lambda_{\mathsf{T}}P=-\mathrm{d}^\Lambda_{\mathsf{T}}(NP)$ and (PN) is not a Poisson-Nijenhuis structure.

It is easy to see that, as in the classical case, a Poisson-Nijenhuis structure for a Lie algebroid induces a whole hierarchy of compatible Poisson structures and Nijenhuis tensors (see [6]). Since this theory goes quite parallel to the classical case, we will not present details here.

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